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A NUMERICAL STUDY FOR NON-LINEAR MULTI-TERM FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this article, a numerical method is presented for solving non-linear multi-order fractional differential equations (FDEs) using orthonormal Boubaker polynomials as basis functions. The operational matrix is constructed for the product of basis functions. The fractional derivative is considered in the Caputo sense. Using this technique, non-linear multi-order FDEs converted into a system of non-linear algebraic equations, which can then be solved numerically. The proposed method is implemented using MATLAB for numerical analysis. Using numerical results, validity, and accuracy of the proposed method is given and compared the results with existing literature.

Keywords: Orthonormal Boubaker polynomials, operational matrix, fractional differential equations, convergence analysis, Riemann-Liouville integration, Caputo derivative.

AMS Subject Classification: 34A08, 65L60.

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1 Introduction

Fractional calculus (FC) is a field of mathematics that deals with arbitrary order differentiation and integration (Podlubny, 1998). In recent years FC has more attention due to its wide range of applications in various scientific and engineering disciplines (Ross, 1977; Jafari et al., 2023; Ma et al., 2016). Fractional calculus deals analysis of systems with memory and non-local behavior, making it a powerful tool in modeling complex phenomena. FC has many applications in the field like diffusion processes, bio-engineering, control theory, finance, image processing. The operational matrix method is one of the effective technique for solving fractional differential equations (FDEs). Operational matrices provide a systematic framework for converting differential equations into algebraic equations, which can then be solved numerically. In recent years using different basis functions researchers have developed efficient numerical methods using operational matrix for solving FDEs. In this article we have developd effective numerical techniques for solving a non-linear multi-order fractional differential equations by using orthonormal Boubaker basis polynomials. In recent years researchers used different basis functions such as the Legendre , Berstein , Fermat, Clique, Chebyshev, Jacobi, and cubic B-spline polynomials to

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construct the operational matrices (Sun et al., 2022; Ali, 2014; Youssri, 2017; Ganji et al., 2021, 2019; Li, 2014). Here, we use the orthonormal Boubaker basis functions, which were initially applied by Boubaker in order to solve the heat equation. These Boubaker polynomials finds many applications in the literature Kashem et al. (2020); Khoso et al. (2020); Bolandtalat et al. (2016); Zhao (2016); Bencheikh et al. (2022).

In this paper, we consider the following type of non-linear multi-order FDEs

$$D_x^{\omega} y(x) + \sum_{j=1}^{n} a_j(x) D_x^{\beta_j} y(x) + \sum_{i=1}^{k} b_i y^i(x) + g(x) = 0, \tag{1}$$

$$y^s(0) = 0, \qquad s = 0, 1, \cdots n-1, \tag{2}$$

where $n-1 < \omega \leq n$, the coefficients $a_j(x)\,(j = 1, 2, \cdots n)$ and $g(x)$ are known functions, $0 < \beta_1 < \beta_2 < \cdots < \beta_n < \omega$, and $D_x^{\omega} y(x)$ is the Caputo fractional derivative of order ω.

The structure of the article is as follows: A few definitions as well as properties of fractional calculus were covered in section 2. Function approximation using orthonormal Boubaker polynomials basis is discussed in section 3. The creation of the operational matrix for the product and fractional integration is covered in section 4. Section 5, is devoted to solving the multi-order FDE class. A few examples are given in section 6 to demonstrate the reliability and suitability of the suggested approach. We have concluded the work of this paper in section 7.

2 Preliminaries

In this section, we review some key definitions of fractional calculus, Boubaker polynomials, which are used throughout the work in this section.

Definition 1. *(See Podlubny (1998)) The Riemann-Liouville (R-L) fractional integral of order ω is defined as*

$$I_x^{\omega} f(x) = \frac{1}{\Gamma(\omega)} \int_a^x (x-\zeta)^{\omega-1} f(\zeta) d\zeta, \qquad \omega \in [n-1, n). \tag{3}$$

where $n \in \mathbb{N}$.

Definition 2. *(See Podlubny (1998)) The Caputo fractional derivative of order ω is defined as*

$$D_x^{\omega} f(x) = I_x^{(n-\omega)} \left(\frac{d^n f(x)}{dx^n} \right) = \frac{1}{\Gamma(n-\omega)} \int_a^x \frac{f^n(\zeta) d\zeta}{(x-\zeta)^{\omega+1-n}}, \tag{4}$$

where $\omega \in (n-1, n)$. If $\omega = n$, then $D_x^{\omega} f(x) = \dfrac{d^n f(x)}{dx^n}$.

Definition 3. *(see Bolandtalat et al. (2016)) The Boubaker polynomials defined as*

$$B_m(x) = \sum_{p=0}^{\zeta(m)} \left[\frac{(m-4p)}{(m-p)} \binom{p}{m-p} \right] \cdot (-1)^p \cdot x^{m-2p}, \tag{5}$$

where $\zeta(m) = \frac{2m+[(-1)^m-1]}{4}$.

The reduction formula for Boubaker polynomials is given by

$$B_n(x) = x B_{n-1}(x) - B_{n-2}(x), \text{ for } n \geq 2, \tag{6}$$

$$B_0(x) = 1, \; B_1(x) = x.$$

Definition 4. *The orthonormal Boubaker polynomials (Kashem et al., 2020) are defined as follows:*

$$\begin{aligned}
B_{o_0}(x) &= 1,\\
B_{o_1}(x) &= \sqrt{3}(-1+2x),\\
B_{o_2}(x) &= \sqrt{5}(1-6x+6x^2),\\
B_{o_3}(x) &= \sqrt{7}(-1+12x-30x^2+20x^3).
\end{aligned}$$

Analytical form of orthonormal Boubaker polynomial is given by,

$$B_{o_m}(x) = \sqrt{2m+1}\sum_{r=0}^{m}(-1)^{m+r}\frac{(m+r)!x^r}{(m-r)!(r!)^2}, \;\; m \in \mathbb{N}. \tag{7}$$

Properties of fractional operators

1.
$$I_x^{\omega} D_x^{\omega} u(x) = u(x) - \sum_{r=0}^{n-1} u^{(r)}(0)\frac{x^r}{r!}.$$

2.
$$I_x^{\omega} D_x^{\beta} u(x) = I_x^{\omega-\beta} u(x) - \sum_{r=0}^{n-1} \frac{u^{(r)}(0)}{\Gamma(\omega-\beta+r+1)}(x-a)^{\omega-\beta+r},$$

where $\beta \in (n-1, n], n \in \mathbb{N}$.

3.
$$I^{\omega} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\omega)} x^{\beta+\omega}.$$

3 Function approximation

The function $y(x) \in L^2[0,1]$ is approximated using orthonormal Boubaker polynomials as follows:

$$y(x) = \sum_{j=0}^{m} c_j B_{o_j}(x) = C^T B_o(x), \tag{8}$$

where

$$B_o(x) = [B_{o_0}(x) \quad B_{o_1}(x) \ldots B_{o_m}(x)].$$

where $B_{o_j}(x),\; j = 0, 1, \cdots m$ are orthonormal Boubaker polynomials and $C^T = [c_0\, c_1 \ldots c_m]$ are orthonormal Boubaker coefficients and m is any positive integer. The c_j is given by,

$$c_j = \int_0^1 y(x) B_{o_j}(x)\, dx.$$

Let us consider, $X1(x)$ be a Taylor's basis and we represent the $B_o(x)$ in orders of Taylor's basis is as follows:

$$B_o(x) = \tilde{Z} X1(x),$$

where $X1(x) = [1 \quad x \quad x^2 \quad \cdots x^m]^T$ and coefficient matrix $\tilde{Z}$ is given by

$$\tilde{Z} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -\sqrt{3} & 2\sqrt{3} & 0 & 0 & \cdots & 0 \\ \sqrt{5} & -6\sqrt{5} & 6\sqrt{5} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^m\sqrt{2m+1} & (-1)^{m+1}\sqrt{2m+1}(m^2+m) & & & & \frac{(-1)^{2m}\sqrt{2m+1}(2m)!}{(m!)^2} \\ \end{bmatrix}. \quad (9)$$

4 Operational matrices for fractional operators

Orthonormality is a crucial factor in solving FDEs through the operational matrix approach. The operational matrices of fractional operators are obtained in this section.

4.1 Operational matrix of integration

Let us evaluate, R-L fractional integration of Boubaker vector $B_o(x)$, see Bolandtalat et al. (2016)

$$I_x^\omega B_o(x) = L^\omega B_o(x), \quad (10)$$

where, L^ω is the $(m+1)\times(m+1)$ operational matrix of fractional integral.
Computation of L^ω is as follows:

$$\begin{aligned} I_x^\omega B_o(x) &= \frac{1}{\gamma(\omega)}\int_0^x (x-\zeta)^{\omega-1} B_o(\zeta)\, d\zeta \\ &= \frac{1}{\gamma(\omega)}\int_0^x (x-\zeta)^{\omega-1} \tilde{Z} X1(\zeta)\, d\zeta \\ &= \tilde{Z}[I_x^\omega 1 \quad I_x^\omega x \quad I_x^\omega x^2 \quad I_x^\omega x^3 \cdots I_x^\omega x^m]^T \\ &= \tilde{Z}\left[\frac{0!}{\Gamma(\omega+1)}x^\omega \quad \frac{1!}{\Gamma(\omega+2)}x^{\omega+1} \quad \cdots \quad \cdots \quad \frac{m!}{\Gamma(\omega+1+m)}x^{\omega+m}\right]^T \\ &= \tilde{Z}\tilde{D}X2(x), \end{aligned} \quad (11)$$

where $X2(x) = [x^\omega \quad x^{\omega+1} \quad x^{\omega+2} \quad x^{\omega+3} \quad \cdots \quad x^{\omega+m}]^T$,

$$\tilde{D} = \begin{bmatrix} \frac{0!}{\Gamma(\omega+1)} & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1!}{\Gamma(\omega+2)} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{2!}{\Gamma(\omega+3)} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & & & & & \frac{m!}{\Gamma(\omega+1+m)} \end{bmatrix}.$$

Now, we expand $X2(x)$ in terms of $B_o(x)$

$$X2(x) = HB_o(x).$$

Let us express, $x^{\omega+i}$ by $m+1$ orders of the orthonormal Boubaker basis

$$x^{\omega+i} \cong h_i B_o(x),$$

where, $h_i = [h_{i,0} \quad h_{i,1} \cdots h_{i,m}]^T$, $h_{i.j} = \int_0^1 x^{\omega+i} B_j(x)\, dx$, $H = [h_0 \quad h_1 \cdots h_m]$.

$$\therefore I^\omega B_o(x) = \tilde{Z}\tilde{D}HB_o(x) = L^\omega B_o(x), \tag{12}$$

where the matrix $L^\omega = \tilde{Z}\tilde{D}H$ is known as operational matrix of integration with respect to the orthonormal Boubaker basis.

4.2 Operational matrix of the product

In this section, we find the operational matrix of the product of the basis functions, see Rostamy et al. (2014). Now, consider the product of basis functions

$$c^T B_o(x) B_o(x)^T \cong B_o(x)^T \hat{C}_p,$$

where $\hat{C}_p$ is called the operational matrix of the product.
Computation of $\hat{C}_p$:

$$C^T B_o(x) B_o(x)^T = C^T B_o(x) X1(x)^T \tilde{Z}^T \tag{13}$$

$$= \left[\sum_{i=0}^{m} c_i B_{o_i}(x), \sum_{i=0}^{m} x c_i B_{o_i}(x), \sum_{i=0}^{m} x^2 c_i B_{o_i}(x), \cdots \sum_{i=0}^{m} x^m c_i B_{o_i}(x) \right] \tilde{Z}^T. \tag{14}$$

Now, we write functions $x^k B_{o_i}(x)$ in terms of orthonormal Boubaker polynomial basis. Thus, we define

$$v_{k,i} = \left[v_{k,i}^0, v_{k,i}^1, v_{k,i}^2 \cdots v_{k,i}^m \right]^T,$$

$$x^k B_{o_i}(x) = \sum_{i=0}^{m} v_{k,i}^i B_{o_i} = v_{k,i} B_o(x),$$

where

$$v_{k,i} = \int_0^1 x^k B_{o_i}(x) B_o(x) dx \tag{15}$$

$$= \left[\int_{i=0}^1 x^k B_{o_i}(x) B_{o_0}(x) dx, \int_{i=0}^1 x^k B_{o_i}(x) B_{o_1}(x) dx, \cdots \int_{i=0}^1 x^k B_{o_i}(x) B_{o_m}(x) dx \right]. \tag{16}$$

Now consider

$$\sum_{i=0}^{m} c_i x^k B_{o_i}(x) = \sum_{i=0}^{m} c_i \left(\sum_{j=0}^{m} v_{k,i}^j B_{o_j}(x) \right) \tag{17}$$

$$= B_o(x)^T \left[v_{k,i}^0, v_{k,i}^1, \cdots v_{k,i}^m \right] C \tag{18}$$

$$= B_o(x)^T V_k C \tag{19}$$

$$= B_o(x)^T \bar{C}, \tag{20}$$

where

$$\bar{C} = \left[V_0 C, V_1 C, \cdots V_m C \right],$$

each V_i is a matrix of order $(m+1) \times (m+1)$

$$C^T B_o(x) B_o(x)^T = C^T B_o(x) X1(x)^T \tilde{Z}^T \tag{21}$$

$$= B_o(x)^T \bar{C} \tilde{Z}^T, \tag{22}$$

where is $\hat{C}_p = \bar{C}\tilde{Z}^T$ is the required operational matrix of the product.

For example: For $m = 3$

$$B_o = \begin{pmatrix} B_{o_0}(x) & B_{o_1}(x) & B_{o_2}(x) & B_{o_3}(x) \end{pmatrix},$$

$$\tilde{Z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\sqrt{3} & 2\sqrt{3} & 0 & 0 \\ \sqrt{5} & -6\sqrt{5} & 6\sqrt{5} & 0 \\ -\sqrt{7} & 12\sqrt{7} & -30\sqrt{7} & 20\sqrt{7} \end{pmatrix},$$

$$X1(x) = \begin{pmatrix} 1 & x & x^2 & x^3 \end{pmatrix}$$

$$V_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad V_1 = \begin{pmatrix} 0.5 & 0.2887 & 0 & 0 \\ 0.2887 & 0.5 & 0.2582 & 0 \\ 0 & 002582 & 0.5 & 0.2535 \\ 0 & 0.2582 & 0.5 & 0.2532 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} 0.3333 & 0.2887 & 0.0745 & 0.0000 \\ 0.2887 & 0.4000 & 0.2582 & 0.0655 \\ 0.0745 & 0.2582 & 0.3810 & 0.2535 \\ 0 & 0.0655 & 0.2535 & 0.5000 \end{pmatrix} \quad V_3 = \begin{pmatrix} 0.25 & 0.2598 & 0.1118 & 0.0189 \\ 0.2598 & 0.3500 & 0.2582 & 0.982 \\ 0.1118 & 0.2490 & 0.3214 & 0.2395 \\ 0.0189 & 0.0982 & 0.2395 & 0.3167 \end{pmatrix}.$$

Using induction in mathematics, we can now approximate $y(x)^k, k \in N$ as follows,

$$y(x) \cong C^T B_o(x).$$

For $k = 2$

$$y(x)^2 \cong C^T B_o(x) B_o(x)^T C \cong B_o^T(x) \hat{C}_p C,$$

where $\hat{C}_p$ is product's operational matrix. Then for $k = 3$, we get

$$y(x)^3 \cong B_o^T(x) \hat{C}_p^{\,2} C.$$

Using induction in mathematics, we can write,

$$y(x)^k = B_o^T(x) \hat{C}_p^{\,k-1} C.$$

5 Solution of non-linear multi-order fractional differential equations

Equation 1 can be rewrite as,

$$D_x^\omega y(x) + \sum_{j=1}^{n} a_j(x) I^{\omega-\beta_j} D_x^\omega y(x) + \sum_{i=1}^{k} b_i (I^\omega D^\omega y(x))^i + g(x) = 0, \tag{23}$$

By considering $D^\omega y(x) = u(x)$, we have

$$u(x) + \sum_{j=1}^{n} a_j(x) I^{\omega-\beta_j} u(x) + \sum_{i=1}^{k} b_i (I^\omega u(x))^i + g(x) = 0, \tag{24}$$

In order to solve equation 1, we approximate $u(x) = D_x^\omega y(x)$ in terms of orthonormal Boubaker polynomials as,

$$D_x^\omega y(x) = \sum_{i=0}^{m} c_i B_{o_i}(x) = C^T B_o(x) \tag{25}$$

(see Behroozifar et al. (2017)).

$$\text{By applying } I^{\omega} \text{ and Using 1, 2 we get } \therefore y(x) = C^T I^{\omega} B_o(x) \cong C^T L^{\omega} B_o(x) = C_{\omega}^T B_o(x)$$

where L^{ω} is the operational matrix of integration and denote $C_{\omega}^T = C^T L^{\omega}$ and $n-1 < \omega \leq n$,. Now we approximate

$$I^{\omega-\beta_j} u(x) = C^T L^{\omega-\beta_j} B_o(x) \tag{26}$$

and $0 < \beta_1 < \beta_2 < \cdots < \beta_n < \omega$. Consider the approximations for $g(x)$ and $a_j(x)$ as follows,

$$g(x) \cong G^T B_o(x), \tag{27}$$

$$a_j(x) \cong \tilde{A}_j^T B_o(x), \tag{28}$$

where the $(m+1) \times 1$ column vectors G and $\tilde{A}_j$ are known, whereas the $(m+1) \times 1$ column vector C is unknown.

Substitute the approximations 25, 26, 27, 28 we get

$$C^T B_o(x) + \sum_{j=1}^{n} \tilde{A}_j^T B_o(x) C^T L^{\omega-\beta_j} B_o(x) + \sum_{i=1}^{k} b_i (C_{\omega}^T B_0(x))^i + G^T B_o(x) = 0 \tag{29}$$

$$B_0^T(x) C + \sum_{j=1}^{n} \tilde{A}_j^T B_o(x) B_0^T(x) (L^{\omega-\beta_j})^T C + \sum_{i=1}^{k} b_i (B_0^T(x) \hat{C_{\omega}}^{i-1} C_{\omega}) + G^T B_o(x) = 0 \tag{30}$$

$$B_0^T(x) C + \sum_{j=1}^{n} B_0^T(x) \hat{A}_j (L^{\omega-\beta_j})^T C + \sum_{i=1}^{k} b_i ((B_0^T(x)) \hat{C_{\omega}}^{i-1} C_{\omega} + B_0^T(x) G = 0 \tag{31}$$

$$B_0^T(x) C + \sum_{j=1}^{n} B_0^T(x) \hat{A}_j (L^{\omega-\beta_j})^T C + \sum_{i=1}^{k} b_i (L^{\omega} (B_0^T(x)) \hat{C_{\omega}}^{i-1} C_{\omega} + B_0^T(x) G = 0 \tag{32}$$

$$C + \sum_{j=1}^{n} \hat{A}_j (L^{\omega-\beta_j})^T C + \sum_{i=1}^{k} b_i (\hat{C_{\omega}})^{i-1} (L^{\omega})^T C + G = 0. \tag{33}$$

where $\hat{A}_j$ and $\hat{C_{\omega}}$ are the operational matrices of the product. Now by solving this non-linear system we get value of C.

6 Numerical Results

Example 1. Consider the following example (Rostamy et al., 2014),

$$D^{2.5} y(x) + D^{1.25} y(x) + y(x) + y^2(x) - y^3(x) = 12\sqrt{\frac{x}{\pi}} + \frac{32 x^{\frac{7}{4}}}{7\Gamma(3/4)} + x^3 + x^6 - x^9, \quad 0 \leq x \leq 1$$

with $y''(0) = 0, y'(0) = 0, y(0) = 0$. The exact solution for this equation is $y(x) = x^3$.

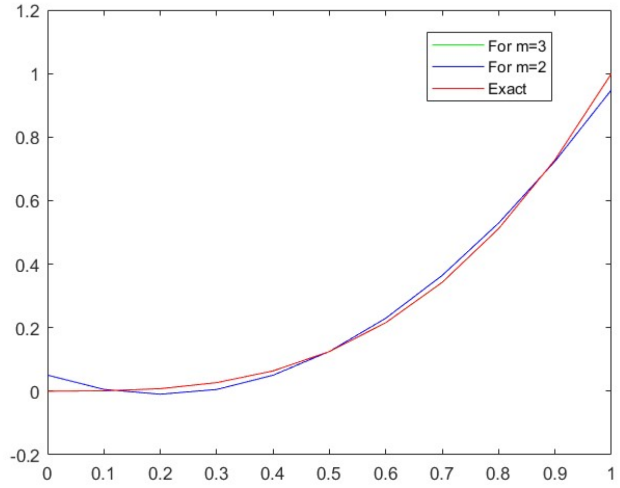

Figure 1: Graphical representation of Exact and approximate solution for m=3 in Example 1

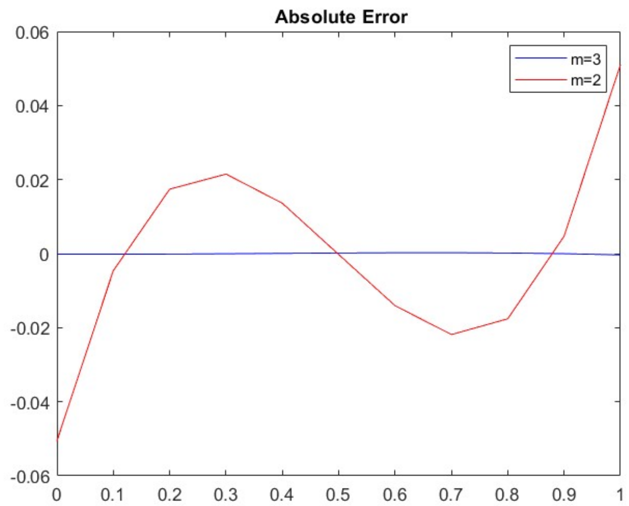

Figure 2: Plot of approximate solution for different value of ω in Example 1

Table 1: Absolute error in example 3 for m=3

x	Our Method	Exact Solution	Absolute Error
0.1	0.001574	0.001000	1.574335e-04
0.2	0.008120	0.008000	1.202248e-04
0.3	0.027035	0.027000	3.533424e-05
0.4	0.063929	0.064000	7.059868e-05
0.5	0.124829	0.125000	1.709344e-04
0.6	0.215760	0.216000	2.390336e-04
0.7	0.342751	0.342999	2.482567e-04
0.8	0.511828	0.512000	1.719641e-04
0.9	0.729016	0.729000	1.648349e-05
1	1.000343	1.000000	3.437257e-04

Example 2. Consider the following example (Yüzbaşı, 2013)

$$D^{\omega}y(x) + y^2(x) = 1, \qquad 0 < \alpha < 1$$

with initial conditions $y'(0) = 0$, $y(0) = 0$. Here this example is solved for $\omega = 1, 0.9, 0.75$ taking $m = 3$.

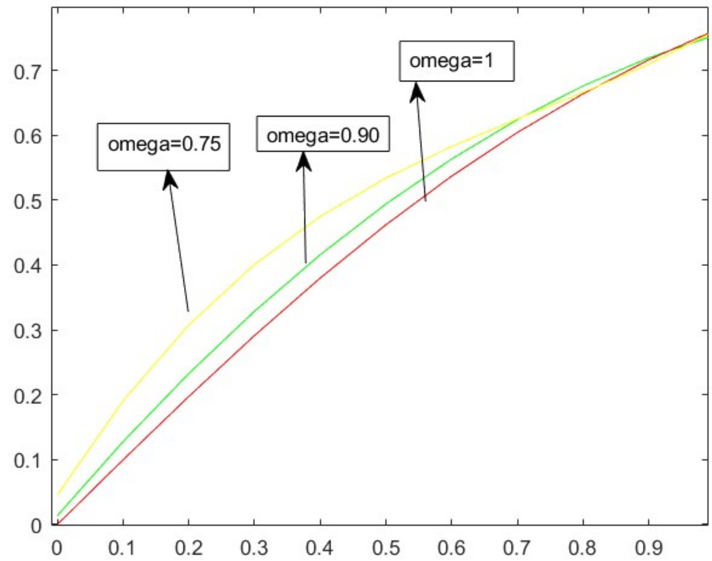

Figure 3: Plot of approximate solution for different value of ω in Example 2

Table 2: Comparison of obtained values of $y(x)$ for Example 2

x	Our Method m=3	Yüzbaşı (2013) m=12	Exact $\omega = 1$	Absolute Error
0.1	0.12760935	0.13003745	0.09966799	0.02794136
0.2	0.23267670	0.23878913	0.19737532	0.03530138
0.3	0.32873788	0.33596217	0.29131261	0.03742527
0.4	0.41585460	0.42258308	0.37994896	0.03590563
0.5	0.49408859	0.49913519	0.46211716	0.03197143
0.6	0.56350156	0.56617156	0.53704957	0.02645199
0.7	0.62415525	0.62439622	0.60436778	0.01978747
0.8	0.67611136	0.67462699	0.66403678	0.01207459
0.9	0.71943163	0.71773475	0.71629788	0.00313376
1.0	0.75417777	0.75458880	0.76159416	0.00741639

Example 3. Consider the following example (Yüzbaşı, 2013),

$$D^{\omega}y(x) = 2y(x) - y^2(x) + 1, \qquad 0 < \omega \leq 1$$

which satisfies $y'(0) = 0$, $y(0) = 0$.

Exact solution for $\omega = 1$ is $y = 1 + \sqrt{2}\tanh(\sqrt{2}x + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1}))$.
This example is solved for $\omega = 1, 0.9, 0.75$ taking $m = 3$.

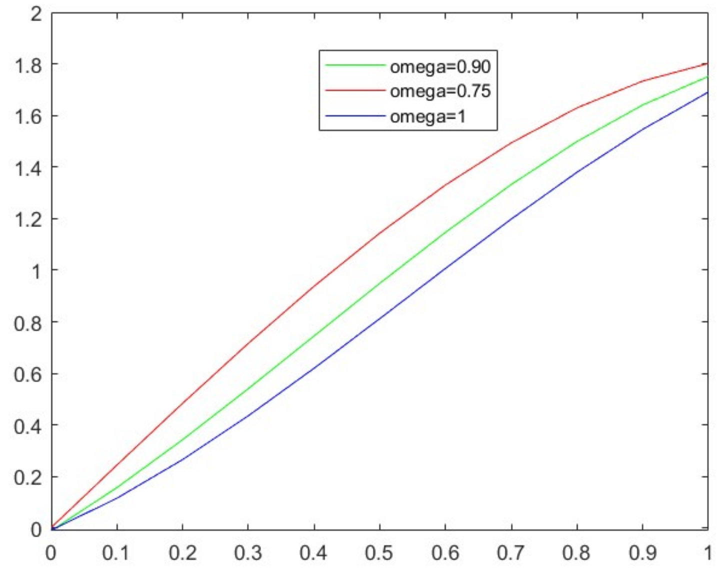

Figure 4: Graphical representation of approximate solution for different value of ω in Example 3

Table 3: Comparison of obtained values of $y(x)$ for Example 3

x	Our method $\omega = 0.90$ m=3	Yüzbaşı (2013) $\omega = 0.90$ m=15	Exact solu $\omega = 1$	Our method $\omega = 1$ m=3	Yüzbaşı (2013) $\omega = 1$ m=30
0.1	0.15730219	0.15070989	0.11029519	0.11623599	0.11029519
0.2	0.34304015	0.31486440	0.24197679	0.26607605	0.24197679
0.3	0.54111482	0.49866532	0.39510484	0.43604079	0.39510484
0.4	0.74531828	0.69753897	0.56781216	0.62020057	0.56781216
0.5	0.94944256	0.90366760	0.75601439	0.81262579	0.75601439
0.6	1.14727972	1.10786162	0.95356621	1.00738682	0.95356621
0.7	1.33262182	1.30143258	1.15294896	1.19855403	1.15294896
0.8	1.49926091	1.47770301	1.34636365	1.38019780	1.34636365
0.9	1.64098905	1.63273978	1.52691131	1.54638852	1.52691131
1.0	1.75159828	1.76527518	1.6894498	1.69119657	1.68949839

7 Conclusion

In this work, a numerical technique has been presented for solving a class of non-linear multi-order fractional differential equations by using orthonormal Boubaker polynomials. Some numerical results are provided to demonstrate the effectiveness of this approach, accompanied by comparisons to previous research using different polynomials. This comparative study used to validate the efficiency and accuracy of the proposed methodology. The computational implementation is carried out using MATLAB software. In future, we can implement this technique to solve coupled FDEs and fractional partial differential equations.

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